

NUMERICAL METHOD FOR THE IDENTIFICATION OF THE LAMÉ COEFFICIENTS IN LINEAR ELASTIC WAVE EQUATION

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Abstract - The purpose of this research is to propose a numerical method for the Lamé coefficients identification problem in linear elastic wave equation: Determine the unknown Lamé coefficients from the knowledge of the plural sets of simultaneous displacement and traction on the surface of an isotropic, linearly elastic bounded body. We assume that the initial displacement and velocity are given in the whole domain. To determine the unknown Lamé coefficients numerically, we adopt the adjoint numerical method. The minimizing problem with some constraints is introduced by using the variational method. We show that the objective functional is Gâteaux differentiable on the appropriate assumptions. To find the minimum, we employ the projected gradient method. The search direction is set by using the Gâteaux derivative of the object functional. We show the efficiency of our method by a simple numerical experiment.

1. INTRODUCTION

In this research, we consider the coefficient identification problem in linear elastic wave equation in two dimensions. We assume that the elastic body is in a plane strain state.

Let $\Omega \subset \mathbf{R}^2$ be a cross section of an isotropic, linearly elastic, bounded body with smooth boundary $\partial\Omega$. We denote by u_i the i -th component of the displacement ($i = 1, 2$), by ε_{ij} and σ_{ij} the ij -th component of the strain and the stress tensors, respectively. The kinematic equations relating the displacement to the strain are described by

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

The constitutive equations representing Hooke's law are given by

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij},$$

where δ_{ij} is Kronecker's delta tensor, in which Einstein's summation convention is used for repeated indices. Here λ and μ are the Lamé coefficients. The equations of motion are given by

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j} \quad \text{in } \Omega \times (0, T], \quad (1)$$

where ρ and T are the density and the duration of observation time, respectively.

We assume that the Lamé coefficients λ and μ belong to $L^\infty(\Omega)$ and satisfy the following conditions:

$$0 < C_\lambda^{(1)} \leq \lambda(\mathbf{x}) \leq C_\lambda^{(2)}, \quad 0 < C_\mu^{(1)} \leq \mu(\mathbf{x}) \leq C_\mu^{(2)} \quad (2)$$

for all $\mathbf{x} \in \Omega$. Here $C_\lambda^{(l)}$ and $C_\mu^{(l)}$ ($l = 1, 2$) are given positive constants. We notice that assumptions (2) are stronger condition compared with the conventional ones [2]. Moreover we assume that the density ρ ,

the initial displacement $u_i(\cdot, 0) = f_i^{(m)}$, and the initial velocity $\frac{\partial u_i^{(m)}}{\partial t}(\cdot, 0) = g_i^{(m)}$ are given in the whole domain. Then, our problem is to determine the Lamé coefficients λ and μ from the knowledge of the plural sets of simultaneous surface displacements $\bar{u}_i^{(m)}$ and tractions $\bar{S}_i^{(m)}$ on $\partial\Omega \times (0, T]$ ($m = 1, 2, \dots, N$). The uniqueness of this inverse problem is guaranteed if the Lamé coefficients are smooth functions and the Dirichlet-Neumann map is given instead of the finite number of boundary measurements [7].

To determine the unknown Lamé coefficients numerically, we make use of the adjoint numerical method [8, 9]. This method is often employed for solving the inverse boundary value problem in control theory [5]. We introduce an objective functional to be minimized, and then the problem is recast as a variational problem. We show that the objective functional is Gâteaux differentiable under certain assumptions. We propose a numerical algorithm based on the projected gradient method in order to find the minimum of the functional. The search direction in this method is given by using the Gâteaux derivative of the objective functional. We confirm the efficiency of our algorithm by a simple numerical experiment.

2. ADJOINT NUMERICAL METHOD

Let $F \subset \Omega$ be a given compact set such that $\partial\Omega \cap F = \emptyset$. We now define a subset \mathcal{K} of $L^\infty(\Omega) \times L^\infty(\Omega)$ as follows:

$$\mathcal{K} = \left\{ (\lambda, \mu) \in L^\infty(\Omega) \times L^\infty(\Omega) \mid \lambda \in C^\infty(\Omega \setminus F), \mu \in C^\infty(\Omega \setminus F), \right. \\ \left. |\nabla\lambda| \leq C_\lambda^{(3)} \text{ on } \Omega \setminus F, |\nabla\mu| \leq C_\mu^{(3)} \text{ on } \Omega \setminus F \right\},$$

where $C_\lambda^{(3)}$ and $C_\mu^{(3)}$ are given positive constants. We denote by $u_i^{(m)}[\lambda, \mu]$ the i -th component of the weak solution to the linear elastic wave eqn.(1) with the Lamé coefficients $(\lambda, \mu) \in \mathcal{K}$ and the surface displacement $\bar{u}_i^{(m)}$. Moreover, we denote by $\varepsilon_{ij}^{(m)}[\lambda, \mu]$ and $\sigma_{ij}^{(m)}[\lambda, \mu]$ the ij -th component of the strain and the stress tensors obtained by using $u_i^{(m)}[\lambda, \mu]$, respectively. The unknown Lamé coefficients are determined by minimizing the functional $J : \mathcal{K} \rightarrow \mathbf{R}_+ := [0, \infty)$, defined as

$$J(\lambda, \mu) = \frac{\bar{\eta}}{N} \sum_{m=1}^N \sum_{i=1}^2 \int_0^T \int_{\partial\Omega} |S_i^{(m)}[\lambda, \mu] - \bar{S}_i^{(m)}|^2 ds dt, \quad (3)$$

where $S_i^{(m)}[\lambda, \mu]$ is the i -th component of the surface traction obtained by using $u_i^{(m)}[\lambda, \mu]$ and $\bar{\eta}$ is a representative speed of the elastic body Ω .

To find the minimum of the functional J , we make use of the projected gradient method [4, 6]: For $l = 0, 1, 2, \dots$,

$$\begin{pmatrix} \lambda_{l+1} \\ \mu_{l+1} \end{pmatrix} = \begin{pmatrix} P_\lambda(\lambda_l - \alpha_l J_\lambda(\lambda_l, \mu_l)) \\ P_\mu(\mu_l - \alpha_l J_\mu(\lambda_l, \mu_l)) \end{pmatrix} \quad (4)$$

where the positive constant α_l is a suitable step size and the maps P_λ and P_μ are the clip-off operators, defined by

$$P_\lambda(\lambda)(\mathbf{x}) := \begin{cases} C_\lambda^{(1)} & (\lambda(\mathbf{x}) < C_\lambda^{(1)}) \\ \lambda(\mathbf{x}) & (C_\lambda^{(1)} \leq \lambda(\mathbf{x}) \leq C_\lambda^{(2)}) \\ C_\lambda^{(2)} & (\lambda(\mathbf{x}) > C_\lambda^{(2)}) \end{cases}$$

and

$$P_\mu(\mu)(\mathbf{x}) := \begin{cases} C_\mu^{(1)} & (\mu(\mathbf{x}) < C_\mu^{(1)}) \\ \mu(\mathbf{x}) & (C_\mu^{(1)} \leq \mu(\mathbf{x}) \leq C_\mu^{(2)}) \\ C_\mu^{(2)} & (\mu(\mathbf{x}) > C_\mu^{(2)}) \end{cases},$$

respectively. The function $J_\lambda(\lambda, \mu)$ represents the first variation of the functional J in the direction to λ , defined by

$$J(\lambda + \delta\lambda, \mu) - J(\lambda, \mu) = \int_\Omega J_\lambda(\lambda, \mu) \delta\lambda dx + o(\|\delta\lambda\|)$$

for any variation $\delta\lambda$ in λ , with a real valued functional $o(\|\delta\lambda\|)$ of higher order than $\|\delta\lambda\|$ as it tends to zero in the norm $\|\varphi\| := \text{ess sup}_{\mathbf{x} \in \Omega} |\varphi(\mathbf{x})|$. The function $J_\mu(\lambda, \mu)$ represents the first variation of the functional J in the direction to μ , defined by

$$J(\lambda, \mu + \delta\mu) - J(\lambda, \mu) = \int_\Omega J_\mu(\lambda, \mu) \delta\mu dx + o(\|\delta\mu\|)$$

for any variation $\delta\mu$ in μ . In order to use this method, we require an expression of the first variations J_λ and J_μ .

We first attempt to get the expression of the first variation $J_\lambda(\lambda, \mu)$. For this purpose, we try to obtain the Gâteaux partial derivative of the functional J in the direction to λ , defined by

$$J_{W,\lambda}(\lambda, \mu)h = \lim_{\zeta \rightarrow 0} \frac{J(\lambda + \zeta h, \mu) - J(\lambda, \mu)}{\zeta}.$$

We notice that the Gâteaux partial derivative $J_{W,\lambda}(\lambda, \mu)$ coincides with the first variation $J_\lambda(\lambda, \mu)$ if $J_{W,\lambda}(\lambda, \mu)$ is bounded linear functional on \mathcal{K} and $J_{W,\lambda}(\lambda, \mu)h$ can be represented as the functional form over $L^\infty(\Omega)$.

For any $(\lambda, \mu) \in \mathcal{K}$, $\zeta \in \mathbf{R}$, and $h \in L^\infty(\Omega)$ such that $(\lambda + \zeta h, \mu) \in \mathcal{K}$, we notice that

$$\begin{aligned} & J(\lambda + \zeta h, \mu) - J(\lambda, \mu) \\ &= \frac{\bar{\eta}}{N} \int_0^T \int_{\partial\Omega} \left\{ 2 \left(S_i^{(m)}[\lambda, \mu] - \bar{S}_i^{(m)} \right) \right\} \delta S_i^{(m)} ds dt + \frac{\bar{\eta}}{N} \sum_{m=1}^N \sum_{i=1}^2 \int_0^T \int_{\partial\Omega} |\delta S_i^{(m)}|^2 ds dt, \end{aligned} \quad (5)$$

where $\delta S_i^{(m)} := S_i^{(m)}[\lambda + \zeta h, \mu] - S_i^{(m)}[\lambda, \mu]$. To analyze the right hand side of (5), we introduce the function $v_i^{(m)}$ which is the i -th component of the weak solution to the initial-boundary value problem

$$\begin{cases} \rho \frac{\partial^2 v_i^{(m)}}{\partial t^2} = \frac{\partial \hat{\sigma}_{ij}^{(m)}}{\partial x_j} & \text{in } \Omega \times [0, T), \\ \hat{\varepsilon}_{ij}^{(m)} = \frac{1}{2} \left(\frac{\partial v_i^{(m)}}{\partial x_j} + \frac{\partial v_j^{(m)}}{\partial x_i} \right) & \text{in } \Omega \times [0, T), \\ \hat{\sigma}_{ij}^{(m)} = 2\mu \hat{\varepsilon}_{ij}^{(m)} + \lambda \hat{\varepsilon}_{kk}^{(m)} \delta_{ij} & \text{in } \Omega \times [0, T), \\ v_i^{(m)} = w_i^{(m)}, \quad \frac{\partial v_i^{(m)}}{\partial t} = 0 & \text{on } \Omega \times \{T\}, \\ v_i^{(m)} = 2 \left(S_i^{(m)}[\lambda, \mu] - \bar{S}_i^{(m)} \right) & \text{on } \partial\Omega \times [0, T), \end{cases} \quad (6)$$

for the "initial" time T . Here $w_i^{(m)}$ is the i -th component of the weak solution to the boundary value problem

$$\begin{cases} \frac{\partial \tilde{\sigma}_{ij}^{(m)}}{\partial x_j} = 0 & \text{in } \Omega, \\ \tilde{\varepsilon}_{ij}^{(m)} = \frac{1}{2} \left(\frac{\partial w_i^{(m)}}{\partial x_j} + \frac{\partial w_j^{(m)}}{\partial x_i} \right) & \text{in } \Omega, \\ \tilde{\sigma}_{ij}^{(m)} = 2\mu \tilde{\varepsilon}_{ij}^{(m)} + \lambda \tilde{\varepsilon}_{kk}^{(m)} \delta_{ij} & \text{in } \Omega, \\ w_i^{(m)} = 2 \left(S_i^{(m)}[\lambda, \mu](\cdot, T) - \bar{S}_i^{(m)}(\cdot, T) \right) & \text{on } \partial\Omega. \end{cases} \quad (7)$$

We now define $\delta\sigma_{ij}^{(m)}$ by $\delta\sigma_{ij}^{(m)} = \sigma_{ij}^{(m)}[\lambda + \zeta h, \mu] - \sigma_{ij}^{(m)}[\lambda, \mu]$. Then, from (5) and (6), we have

$$J(\lambda + \zeta h, \mu) - J(\lambda, \mu) = \frac{\bar{\eta}}{N} \int_0^T \int_{\partial\Omega} v_i^{(m)} \delta\sigma_{ij}^{(m)} n_j ds dt + \frac{\bar{\eta}}{N} \sum_{m=1}^N \sum_{i=1}^2 \int_0^T \int_{\partial\Omega} |\delta S_i^{(m)}|^2 ds dt. \quad (8)$$

From the integration by part with respect to the space direction, we can obtain

$$\begin{aligned} \int_0^T \int_{\partial\Omega} v_i^{(m)} \delta\sigma_{ij}^{(m)} n_j ds dt &= \int_0^T \int_{\Omega} \hat{\varepsilon}_{ij}^{(m)} \delta\sigma_{ij}^{(m)} dx dt + \int_0^T \int_{\Omega} v_i^{(m)} \frac{\partial \delta\sigma_{ij}^{(m)}}{\partial x_j} dx dt \\ &= \int_0^T \int_{\Omega} \hat{\varepsilon}_{ij}^{(m)} \delta\sigma_{ij}^{(m)} dx dt + \int_0^T \int_{\Omega} v_i^{(m)} \rho \frac{\partial^2 \delta u_i^{(m)}}{\partial t^2} dx dt \end{aligned} \quad (9)$$

for each m , where $\delta u_i^{(m)} := u_i^{(m)}[\lambda + \zeta h, \mu] - u_i^{(m)}[\lambda, \mu]$. From Hooke's law, we can obtain the following relation:

$$\delta\sigma_{ij}^{(m)} = \zeta h_\lambda \varepsilon_{kk}^{(m)}[\lambda, \mu] \delta_{ij} + \zeta h_\lambda \delta\varepsilon_{kk}^{(m)} \delta_{ij} + 2\mu \delta\varepsilon_{ij}^{(m)} + \lambda \delta\varepsilon_{kk}^{(m)} \delta_{ij}.$$

Therefore, we have

$$\begin{aligned} \delta\sigma_{ij}^{(m)} \hat{\varepsilon}_{ij}^{(m)} &= \zeta h_\lambda \varepsilon_{kk}^{(m)}[\lambda, \mu] \hat{\varepsilon}_{ij}^{(m)} \delta_{ij} + \zeta h_\lambda \delta\varepsilon_{kk}^{(m)} \hat{\varepsilon}_{ij}^{(m)} \delta_{ij} + 2\mu \delta\varepsilon_{ij}^{(m)} \hat{\varepsilon}_{ij}^{(m)} + \lambda \delta\varepsilon_{kk}^{(m)} \hat{\varepsilon}_{ij}^{(m)} \delta_{ij} \\ &= \zeta h_\lambda \varepsilon_{kk}^{(m)}[\lambda, \mu] \hat{\varepsilon}_{pp}^{(m)} + \zeta h_\lambda \delta\varepsilon_{kk}^{(m)} \hat{\varepsilon}_{pp}^{(m)} + 2\mu \delta\varepsilon_{ij}^{(m)} \hat{\varepsilon}_{ij}^{(m)} + \lambda \delta\varepsilon_{ij}^{(m)} \hat{\varepsilon}_{pp}^{(m)} \delta_{ij} \\ &= \zeta h_\lambda \varepsilon_{kk}^{(m)}[\lambda, \mu] \hat{\varepsilon}_{pp}^{(m)} + \zeta h_\lambda \delta\varepsilon_{kk}^{(m)} \hat{\varepsilon}_{pp}^{(m)} + \delta\varepsilon_{ij}^{(m)} \left(2\mu \hat{\varepsilon}_{ij}^{(m)} + \lambda \hat{\varepsilon}_{pp}^{(m)} \delta_{ij} \right) \\ &= \zeta h_\lambda \varepsilon_{kk}^{(m)}[\lambda, \mu] \hat{\varepsilon}_{pp}^{(m)} + \zeta h_\lambda \delta\varepsilon_{kk}^{(m)} \hat{\varepsilon}_{pp}^{(m)} + \delta\varepsilon_{ij}^{(m)} \hat{\sigma}_{ij}^{(m)}. \end{aligned} \quad (10)$$

By substituting (10) into (9), we obtain

$$\begin{aligned} \int_0^T \int_{\partial\Omega} v_i^{(m)} \delta\sigma_{ij}^{(m)} n_j ds dt &= \zeta \int_0^T \int_{\Omega} h \varepsilon_{kk}^{(m)} [\lambda, \mu] \widehat{\varepsilon}_{pp}^{(m)} dx dt + \zeta \int_0^T \int_{\Omega} h \delta\varepsilon_{kk}^{(m)} \widehat{\varepsilon}_{pp}^{(m)} dx dt \\ &+ \int_0^T \int_{\Omega} \delta\varepsilon_{ij}^{(m)} \widehat{\sigma}_{ij}^{(m)} dx dt + \int_0^T \int_{\Omega} \rho \frac{\partial^2 \delta u_i^{(m)}}{\partial t^2} v_i^{(m)} dx dt. \end{aligned} \quad (11)$$

Using the integration by part and $\delta u_i^{(m)}|_{\partial\Omega \times (0, T]} = 0$, we obtain

$$\int_0^T \int_{\Omega} \delta\varepsilon_{ij}^{(m)} \widehat{\sigma}_{ij}^{(m)} dx dt = - \int_0^T \int_{\Omega} \delta u_i^{(m)} \rho \frac{\partial^2 v_i^{(m)}}{\partial t^2} dx dt. \quad (12)$$

Since $\delta u_i^{(m)}(\cdot, 0) = 0$, $\frac{\partial \delta u_i^{(m)}}{\partial t}(\cdot, 0) = 0$, $v_i^{(m)}(\cdot, T) = w_i^{(m)}$, and $\frac{\partial v_i^{(m)}}{\partial t}(\cdot, T) = 0$ in Ω , we obtain the relation

$$\int_0^T \int_{\Omega} \left\{ \rho \frac{\partial^2 \delta u_i^{(m)}}{\partial t^2} v_i^{(m)} - \delta u_i^{(m)} \rho \frac{\partial^2 v_i^{(m)}}{\partial t^2} \right\} dx dt = \int_{\Omega} \rho \frac{\partial \delta u_i^{(m)}}{\partial t}(\cdot, T) w_i^{(m)} dx. \quad (13)$$

From eqns (11)–(13), we have

$$\begin{aligned} \int_0^T \int_{\partial\Omega} v_i^{(m)} \delta\sigma_{ij}^{(m)} n_j ds dt \\ = \zeta \int_0^T \int_{\Omega} h \varepsilon_{kk}^{(m)} [\lambda, \mu] \widehat{\varepsilon}_{pp}^{(m)} dx dt + \int_{\Omega} \rho \frac{\partial \delta u_i^{(m)}}{\partial t}(\cdot, T) w_i^{(m)} dx + \zeta \int_0^T \int_{\Omega} h \delta\varepsilon_{kk}^{(m)} \widehat{\varepsilon}_{pp}^{(m)} dx dt. \end{aligned} \quad (14)$$

Hence, from (8) and (14), we can obtain the following relation:

$$\begin{aligned} J(\lambda + \zeta h, \mu) - J(\lambda, \mu) &= \zeta \int_{\Omega} h \left(\frac{\bar{\eta}}{N} \int_0^T \varepsilon_{kk}^{(m)} [\lambda, \mu] \widehat{\varepsilon}_{pp}^{(m)} dt \right) dx + \frac{\bar{\eta}}{N} \int_{\Omega} \rho \frac{\partial \delta u_i^{(m)}}{\partial t}(\cdot, T) w_i^{(m)} dx \\ &+ \zeta \frac{\bar{\eta}}{N} \int_0^T \int_{\Omega} h \delta\varepsilon_{kk}^{(m)} \widehat{\varepsilon}_{pp}^{(m)} dx dt + \frac{\bar{\eta}}{N} \sum_{m=1}^N \sum_{i=1}^2 \int_0^T \int_{\partial\Omega} |\delta S_i^{(m)}|^2 ds dt. \end{aligned}$$

We assume that the density ρ is a positive constant in the whole domain and the surface displacements $\bar{u}_i^{(m)} \in C^6([0, T]; H^{\frac{5}{2}}(\partial\Omega))$ with $\frac{\partial^k \bar{u}_i^{(m)}}{\partial t^k}(\cdot, 0) = 0$ for $k = 0, 1, \dots, 5$. Then, we can get the Gâteaux derivative of J by using Choi and Nakamura's result in the scalar wave case [3]. They have obtained the Gâteaux derivative of the functional

$$V \ni K \mapsto \int_0^T \int_{\partial\Omega} |K \frac{\partial u}{\partial n} - \bar{q}|^2 ds dt,$$

where u is a solution to the scalar wave equation

$$\frac{\partial^2 u}{\partial t^2} - \nabla \cdot (K \nabla u) = 0 \quad \text{in } \Omega \times (0, T]$$

and

$$V := \left\{ K \in L^\infty(\Omega) \mid 0 < C_1 \leq K \leq C_2 \text{ on } \bar{\Omega}, K \in C^\infty(\Omega \setminus F), |\nabla K| \leq C_3 \text{ on } \Omega \setminus F \right\}.$$

Here C_i ($i = 1, 2, 3$) are given positive constants. We notice that the equations of motion (1) and the objective functional (3) are the same type of Choi and Nakamura's ones. Therefore, by using their technique to estimate the functional, we can obtain

$$\begin{aligned} \lim_{\zeta \rightarrow 0} \frac{1}{\zeta} \int_{\Omega} \rho \frac{\partial \delta u_i^{(m)}}{\partial t}(\cdot, T) w_i^{(m)} dx &= \int_{\Omega} \rho \frac{\partial U_i^{(m)}}{\partial t}[h, 0](\cdot, T) w_i^{(m)} dx, \\ \lim_{\zeta \rightarrow 0} \int_0^T \int_{\Omega} h \delta\varepsilon_{kk}^{(m)} \widehat{\varepsilon}_{pp}^{(m)} dx dt &= 0, \quad \lim_{\zeta \rightarrow 0} \frac{1}{\zeta} \sum_{m=1}^N \sum_{i=1}^2 \int_0^T \int_{\partial\Omega} |\delta S_i^{(m)}|^2 ds dt = 0, \end{aligned}$$

and the map $h \mapsto \int_{\Omega} \rho \frac{\partial U_i^{(m)}}{\partial t} [h, 0](\cdot, T) w_i^{(m)} dx$ is bounded linear on $L^\infty(\Omega)$ for $(\lambda, \mu) \in \mathcal{K}$. Here we denote by the function $U_i^{(m)}[p, q]$ the weak solution of the initial-boundary value problem with the source term $F_i^{(m)}[p, q]$ as follows:

$$\begin{cases} \rho \frac{\partial^2 U_i^{(m)}}{\partial t^2} = \frac{\partial \check{\sigma}_{ij}^{(m)}}{\partial x_j} + F_i^{(m)}[p, q] & \text{in } \Omega \times (0, T], \\ \check{\varepsilon}_{ij}^{(m)} = \frac{1}{2} \left(\frac{\partial U_i^{(m)}}{\partial x_i} + \frac{\partial U_j^{(m)}}{\partial x_j} \right) & \text{in } \Omega \times (0, T], \\ \check{\sigma}_{ij}^{(m)} = 2\mu \check{\varepsilon}_{ij}^{(m)} + \lambda \check{\varepsilon}_{ij}^{(m)} \delta_{ij} & \text{in } \Omega \times (0, T], \\ U_i^{(m)} = 0, \quad \frac{\partial U_i^{(m)}}{\partial t} = 0 & \text{on } \Omega \times \{0\}, \\ U_i^{(m)} = 0 & \text{on } \partial\Omega \times (0, T], \end{cases} \quad (15)$$

where $F_i^{(m)}[p, q] := \frac{\partial \tau_{ij}^{(m)}}{\partial x_j} [p, q]$ and $\tau_{ij}[p, q] := 2q \varepsilon_{ij}^{(m)}[\lambda, \mu] + p \varepsilon_{kk}^{(m)}[\lambda, \mu] \delta_{ij}$.

Therefore, we have

$$J_{W,\lambda}(\lambda, \mu)h = \int_{\Omega} h \left(\frac{\bar{\eta}}{N} \int_0^T \varepsilon_{kk}^{(m)}[\lambda, \mu] \check{\varepsilon}_{pp}^{(m)} dt \right) dx + \frac{\bar{\eta}}{N} \int_{\Omega} \rho \frac{\partial U_i^{(m)}}{\partial t} [h, 0](\cdot, T) w_i^{(m)} dx.$$

Moreover, by Example 5 on page 118 of [10], there exists a unique $s_\lambda \in L^1(\Omega)$ such that

$$\int_{\Omega} h s_\lambda dx = \int_{\Omega} \rho \frac{\partial U_i^{(m)}}{\partial t} [h, 0](\cdot, T) w_i^{(m)} dx \quad \text{for } \forall h \in L^\infty(\Omega). \quad (16)$$

Hence we can obtain the expression of first variation as follows:

$$J_\lambda(\lambda, \mu) = \frac{\bar{\eta}}{N} \left(\int_0^T \varepsilon_{kk}^{(m)} \check{\varepsilon}_{pp}^{(m)} dt + s_\lambda \right).$$

In a similar way, we can determine the Gâteaux partial derivative of the functional J in the direction to μ , defined by

$$J_{W,\mu}(\lambda, \mu)h = \lim_{\zeta \rightarrow 0} \frac{J(\lambda, \mu + \zeta h) - J(\lambda, \mu)}{\zeta}.$$

For any $(\lambda, \mu) \in \mathcal{K}$, $\zeta \in \mathbf{R}$, and $h \in L^\infty(\Omega)$ such that $(\lambda, \mu + \zeta h) \in \mathcal{K}$, we can obtain

$$J_{W,\mu}(\lambda, \mu)h = \int_{\Omega} h \left(\frac{\bar{\eta}}{N} \int_0^T 2\varepsilon_{ij}^{(m)}[\lambda, \mu] \check{\varepsilon}_{ij}^{(m)} dt \right) dx + \frac{\bar{\eta}}{N} \int_{\Omega} \rho \frac{\partial U_i^{(m)}}{\partial t} [0, h](\cdot, T) w_i^{(m)} dx,$$

and there exists a unique $s_\mu \in L^1(\Omega)$ such that

$$\int_{\Omega} h s_\mu dx = \int_{\Omega} \rho \frac{\partial U_i^{(m)}}{\partial t} [0, h](\cdot, T) w_i^{(m)} dx \quad \text{for } \forall h \in L^\infty(\Omega). \quad (17)$$

Hence we have

$$J_\mu(\lambda, \mu) = \frac{\bar{\eta}}{N} \left(\int_0^T 2\varepsilon_{ij}^{(m)}[\lambda, \mu] \check{\varepsilon}_{ij}^{(m)} dt + s_\mu \right).$$

Thus in order to identify the unknown Lamé coefficients, we can summarize the algorithm as follows:

Numerical algorithm

1. Select the initial Lamé coefficients λ_0 and μ_0 which satisfy condition (2).
2. For $l = 0, 1, 2, \dots$; do

- (a) Solve the linear elastic wave eqns (1) with the surface displacements $\bar{u}_i^{(m)}$ to find $\varepsilon_{ij}^{(m)}$ and $S_i^{(m)}$ for $m = 1, 2, \dots, N$.
- (b) Solve the boundary value problems (7) to find $w_i^{(m)}$ for $m = 1, 2, \dots, N$.
- (c) Solve the initial-boundary value problems (6) to find $\hat{\varepsilon}_{ij}^{(m)}$ for $m = 1, 2, \dots, N$.
- (d) Find the function s_λ and s_μ by solving (16) and (17), respectively.
- (e) Calculate the first variations $J_\lambda(\lambda_l, \mu_l)$ and $J_\mu(\lambda_l, \mu_l)$:

$$J_\lambda(\lambda_l, \mu_l) = \frac{\bar{\eta}}{N} \left(\int_0^T \varepsilon_{kk}^{(m)} \hat{\varepsilon}_{pp}^{(m)} dt + s_\lambda \right), \quad J_\mu(\lambda_l, \mu_l) = \frac{\bar{\eta}}{N} \left(\int_0^T 2 \varepsilon_{ij}^{(m)} \hat{\varepsilon}_{ij}^{(m)} dt + s_\mu \right).$$

- (f) Choose the step size α_l .
- (g) Update the Lamé coefficients by (4).

3. NUMERICAL EXPERIMENT

In this section, we show a simple numerical experiment for our algorithm. Let Ω be a disk with the radius of $L = 1$ [m] and assume $\rho = 10.0 \times 10^3$ [kg/m³]. The exact Lamé coefficients λ [Pa] and μ [Pa] are set as follows (see Figures 1 and 2):

$$\lambda(\mathbf{x}) = \begin{cases} 1.15 \times 10^{11} & (\|\mathbf{x}\|_\infty < 0.15) \\ 1.51 \times 10^{11} & (\text{otherwise}) \end{cases},$$

$$\mu(\mathbf{x}) = \begin{cases} 0.77 \times 10^{11} & (\|\mathbf{x}\|_\infty < 0.15) \\ 0.65 \times 10^{11} & (\text{otherwise}) \end{cases},$$

where $\|\cdot\|_\infty$ means the maximum norm of \mathbf{R}^2 .

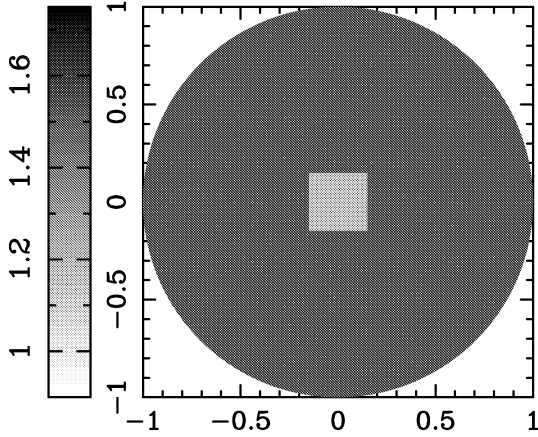


Figure 1. Exact λ .

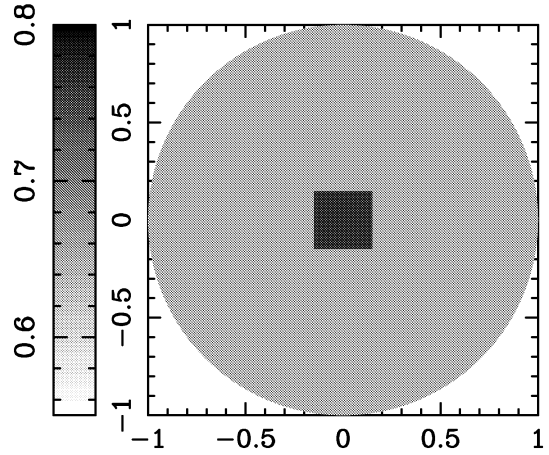


Figure 2. Exact μ .

We utilize the speed of transverse wave on the boundary as the representative speed $\bar{\eta}$, namely,

$$\bar{\eta} = \sqrt{\frac{\mu}{\rho}} \Big|_{\partial\Omega} = 2.55 \times 10^3 \text{ [m/s]}.$$

The constants in the constrained condition (2) are given by $C_\lambda^{(1)} = 1.00 \times 10^{11}$, $C_\lambda^{(2)} = 1.75 \times 10^{11}$, $C_\mu^{(1)} = 0.55 \times 10^{11}$, and $C_\mu^{(2)} = 0.90 \times 10^{11}$. The length of time is $T = 1.01 \times 10^{-3}$ [s] corresponding to $2.6/\bar{\eta}$. The initial displacement and velocity are both set to 0.0. We assume that the number of boundary measurement sets is $N = 3$.

The boundary data for this example are generated by solving numerically the linear elastic problems with the traction $\bar{S}_i^{(m)}|_{\partial\Omega_m \times (0, T]} = -p(t)n_i$ and $\bar{S}_i^{(m)}|_{(\partial\Omega \setminus \partial\Omega_m) \times (0, T]} = 0.0$, where

$$\partial\Omega_m = \left\{ (\cos \theta, \sin \theta) \mid -\frac{\pi}{50} < \theta - (m-1)\frac{\pi}{4} < \frac{\pi}{50} \right\}$$

and

$$p(t) = \begin{cases} \sin\left(\frac{12.5\pi\bar{\eta}}{L}t\right) & \left(0 \leq t \leq \frac{0.16L}{\bar{\eta}}\right) \\ 0.0 & \left(t > \frac{0.16L}{\bar{\eta}}\right) \end{cases} .$$

In order to solve this problem numerically, we make use of the Newmark method [1] for time integration with linear triangular finite elements in space. Calculated values on the circle of the radius $L = 1$ are used for the surface displacements and tractions in this example. The initial-boundary value problems in our algorithm are also solved numerically by using the Newmark method for time integration with linear triangular finite elements in space (see Figure 3).

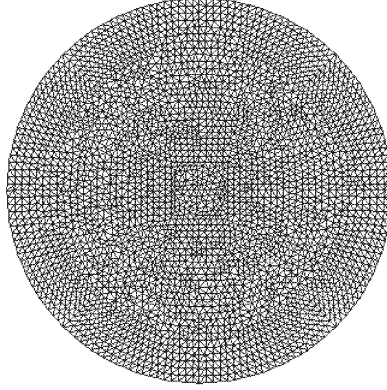


Figure 3. Finite elements (5884FE).

We need to obtain the functions s_λ and s_μ which satisfy the relations (16) and (17) in order to determine the first variation. We adopt the Galerkin method to get approximately these functions. Let $\{B_p\}_{p=1}^{N_B}$ be a set of subsets of Ω such that

$$\bar{\Omega} = \bigcup_{p=1}^{N_B} \bar{B}_p, \quad B_p \cap B_q = \emptyset \quad (p \neq q) .$$

We denote by χ_p the characteristic function satisfying the relation

$$\chi_p(\mathbf{x}) = \begin{cases} 1 & (\mathbf{x} \in B_p) \\ 0 & (\text{otherwise}) \end{cases} .$$

Then the approximated functions $\hat{s}_\lambda = \sum_{p=1}^{N_B} s_{\lambda,p} \chi_p$ and $\hat{s}_\mu = \sum_{p=1}^{N_B} s_{\mu,p} \chi_p$ are obtained by solving the equations

$$\begin{aligned} \int_{\Omega} \chi_p \hat{s}_\lambda dx &= \int_{\Omega} \rho \frac{\partial U_i^{(m)}}{\partial t} [\chi_p, 0](\cdot, T) w_i^{(m)} dx , \\ \int_{\Omega} \chi_p \hat{s}_\mu dx &= \int_{\Omega} \rho \frac{\partial U_i^{(m)}}{\partial t} [0, \chi_p](\cdot, T) w_i^{(m)} dx \end{aligned}$$

for $p = 1, 2, \dots, N_B$, respectively. From the orthogonal relation of the characteristic functions, we notice that

$$\begin{aligned} s_{\lambda,p} &= \frac{1}{|B_p|} \int_{\Omega} \rho \frac{\partial U_i^{(m)}}{\partial t} [\chi_p, 0](\cdot, T) w_i^{(m)} dx , \\ s_{\mu,p} &= \frac{1}{|B_p|} \int_{\Omega} \rho \frac{\partial U_i^{(m)}}{\partial t} [0, \chi_p](\cdot, T) w_i^{(m)} dx \end{aligned}$$

for $p = 1, 2, \dots, N_B$. In this example, we set $N_B = 5$ and

$$B_p = \{\mathbf{x} \in \Omega \mid 0.2(p-1) < \|\mathbf{x}\|_2 < 0.2p\},$$

where $\|\cdot\|_2$ means the Euclidean norm of \mathbf{R}^2 .

We assume that $\lambda_0 = 1.51 \times 10^{11}$ and $\mu_0 = 0.65 \times 10^{11}$ in the whole domain. After 60 iterations, we have the calculated λ_{60} and μ_{60} as shown in Figure 4 and Figure 6, respectively. Figure 5 shows the distribution of the relative error for λ_{60} . The distribution of the relative error for μ_{60} is obtained as shown in Figure 7. These figures show that the estimated distribution of the coefficients are in good agreement with the exact ones. However, the identified values of the coefficients are not satisfactory. From this result, we know that our algorithm must be modified in order to achieve a high resolution for the inverse problem.

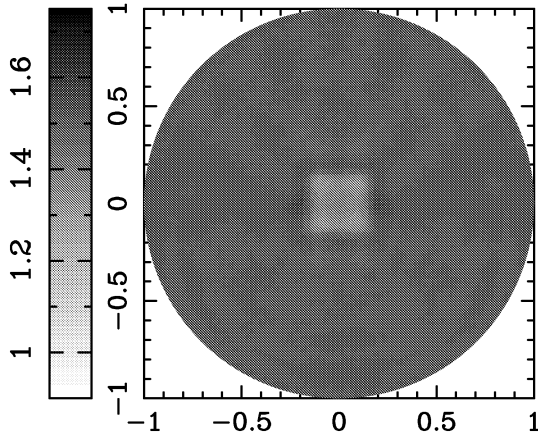


Figure 4. Calculated λ_{60} .

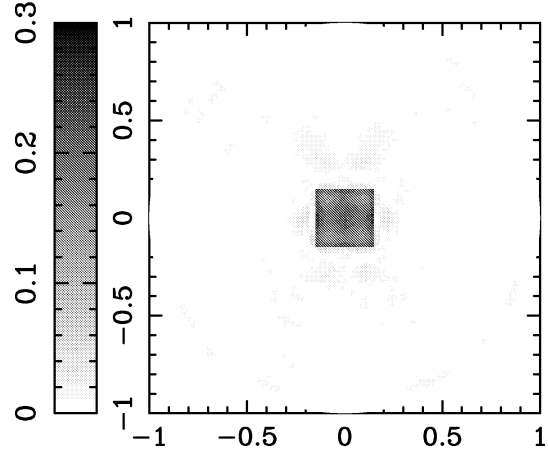


Figure 5. Relative error (coefficient λ).

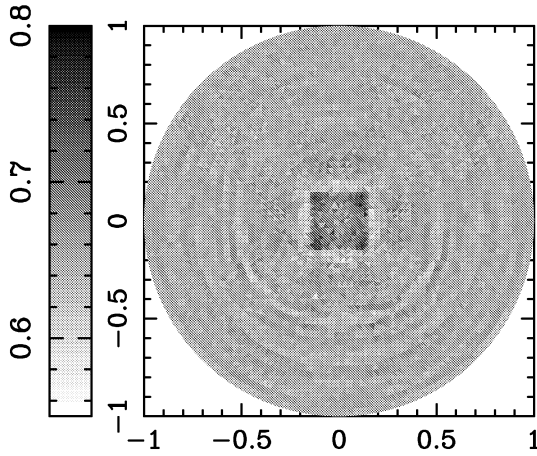


Figure 6. Calculated μ_{60} .

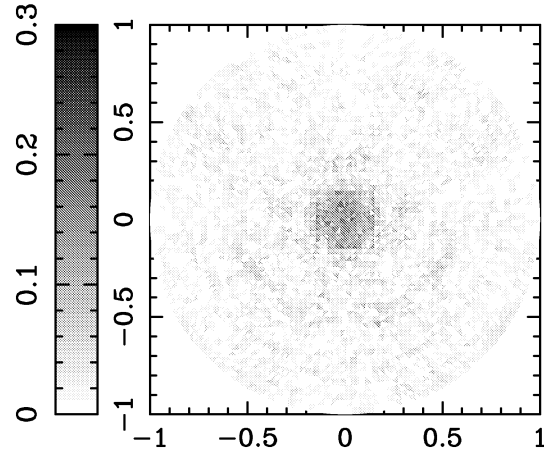


Figure 7. Relative error (coefficient μ).

4. CONCLUDING REMARKS

In this study, we considered the numerical algorithm for the problem of coefficient identification in linear elastic wave equation in two dimensions. The measured data are assumed to be given by the plural sets of simultaneous surface displacements and tractions on the whole boundary. We assume that the density is known, whilst the Lamé coefficients are unknown. To identify numerically the unknown Lamé coefficients, we make use of the adjoint numerical method. The problem is reformulated as a minimization of the functional of two variables with constraints. We show that the objective functional is Gâteaux differentiable with respect to each coefficient. In order to find numerically the minimum of the functional, the algorithm based on the projected gradient method is proposed. The search direction is obtained by using the Gâteaux partial derivative. By a simple numerical experiment, we confirm the efficiency of our algorithm. The distribution of coefficients can be identified by using our algorithm, but the calculated values are not satisfactory. We must modify our algorithm to achieve a high resolution

for our problem. Moreover, we have not obtained the convergence of our algorithm which is deferred to a future investigation.

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